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States and Homomorphisms on the Pták Sum

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Summing of a Boolean algebra and a quantum logic has been defined by P. Pták and studied by, e.g., V. Janiš, Z. Riečanová, O. Nánásiová, and C. A. Drossos. It was shown that there is a special case when this structure is a direct product. Drossos has studied the connection between this structure and a Boolean power. In this paper we investigate the conditions when the Pták sum is a free product and when the connection is between the center of L and the structure of states on $B \oplus L$.

1. INTRODUCTION

Let L be a *quantum logic*. In this paper we consider the quantum logic as an *orthomodular lattice*. Precisely, L is a partially ordered set with the first and the last elements *0* and *1*, respectively, with the orthocomplementation $\perp$: $L \to L$ such that

(1) $(a^{\perp})^{\perp} = a$ for $a \in L$.
(2) $a \leq b$ implies $a^{\perp} \geq b^{\perp}$, where $a, b \in L$.
(3) For all $a \in L$ we have $a^{\perp} \vee a = 1$.
(4) For any $a_1, \ldots, a_n \in L$ there exists $\bigvee_{i=1}^{n} a_i \in L$.
(5) If $a \leq b$, then $b = a \vee (b \wedge a^{\perp})$ $(a, b \in L)$.

Two elements $a, b \in L$ are *orthogonal* if $a \leq b^{\perp}$, and $a, b \in L$ are *compatible* $(a \leftrightarrow b)$ if $a = (a \vee b) \wedge (a \vee b^{\perp})$. If $a_i \in L$ for any $i = 1, 2, 3, 4, \ldots, n$ and $b \in L$ is such that $b \leftrightarrow a_i$ for all i, then $b \leftrightarrow \bigvee_{i=1}^{n} a_i$ and $b \wedge (\bigvee_{i=1}^{n} a_i) = \bigvee_{i=1}^{n} (a_i \wedge b)$ (Varadarajan, 1968).

A subset $L_0 \subseteq L$ is a *sublogic* of L if for any $a \in L_0$ we have $a^{\perp} \in L_0$ and for any $a_1, \ldots, a_n \in L_0$, $\bigvee_{i=1}^{n} a_i \in L_0$. If for any $a, b \in L$, $a \leftrightarrow b$, then L is a *Boolean algebra*. In the following we shall pick up $C(L)$, the *center* of L

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$[C(L) = \{a \in L;\ a \leftrightarrow b$ for any $b \in L$ (Varadarajan, 1968; Pták and Pulmannová, 1989).

A *state* m on L is the map from L to the interval [0, 1] on the real line such that (i) $m(1) = 1$; (ii) $m(\bigvee_{i=1}^{n} a_i) = \sum_{i=1}^{n} m(a_i)$ if $a_i \leq a_j^{\perp}$ for all $a_i \neq a_j$ $(i, j = 1, 2, \ldots, n)$. If L is a quantum logic, then $S(L)$ will be the set of all states on L. For $S \subseteq S(L)$ we shall say that (L, S) is a *quite full system* (qfs) if $\{m \in S: m(a) = 1\} \subseteq \{m \in S: m(b) = 1\}$ implies $a \leq b$ (Pták and Pulmannová, 1989).

Let L_1, L_2 be some logics. Then a mapping $f: L_1 \to L_2$ is called a *homomorphism* if

(1) $f(a^{\perp}) = f(a)^{\perp}$.
(2) $f(a \vee b) = f(a) \vee f(b)$ for a, b from L_1 such that $a \leq b^{\perp}$.

The set $R(f) = \{f(a); a \in L_1\}$ is called the range of homomorphism f. Two homomorphisms $h: L_1 \to L_3$, $g: L_2 \to L_3$ are called *compatible* if for any $a \in L_1$ and for any $b \in L_2$, $h(a) \leftrightarrow g(b)$ (where L_1, L_2, L_3 are quantum logics).

If a mapping $f: L_1 \to L_2$ is injective homomorphism and f^{-1} is homomorphism, then f is called an *embedding* (Pták and Pulmannová, 1989).

Let L, Q be some quantum logics. Let m and h be a state on L and a homomorphism from Q to L, respectively. It is clear that a map m_h from Q to L such that $m_h(a) = m(h(a))$ is a state on Q.

Definition 1.1 (Pulmannová, 1988). Let $(L_1, 0_i, 1_i, {}^{\perp}{}_i)$, $i \in I$, be a set of ortholattices. An ortholattice $(\mathscr{L}, 0, 1, {}^{\perp})$ is a *free product* of the ortholattices L_i, $i \in I$, if:

(i) For any $i \in I$, there is an injective homomorphism $u_i: L_1 \to \mathscr{L}$ preserves the lattice operations and orthocomplementation so that each L_i can be considered as a subalgebra of $\mathscr{L}$, and for $i, j \in I$, $i \neq j$, $L_i - \{0_i, 1_i\}$ are disjoint.

(ii) $\mathscr{L}$ is generated by $\bigcup_i \{u_i(L_i): i \in I\}$.

(iii) For any ortholattice A and for a family of homomorphisms $\phi_i: L_i \to A$, $i \in I$, there exists a homomorphism $\phi: \mathscr{L} \to A$ such that $\phi \circ u_i$ agrees with ϕ_i for all $i \in I$.

Definition 1.2 (Pták, 1986). Let B and L_1 be a Boolean algebra, and a quantum logic, respectively. Then $B \oplus L_1$ is quantum logic with the following properties:

(1) There exist embeddings $f: b \to L$, $f_1: L_1 \to L$ such that $f(a) \wedge f_1(b) = 0$ iff $a = 0$ or $b = 0$.
(2) There is no proper sublogic L containing $f(B) \cup f_1(L_1)$.
(3) For each couple of states $m_0 \in S(B)$, $m_1 \in S(L_1)$ there exists a state $\mu \in S(B \oplus L_1)$ such that $\mu(f(a)) = m_0(a)$ for each $a \in B$ and $\mu(f_1(b)) = m_1(b)$ for any $b \in L_1$ $[\mu = (m_0, m_1)]$.

This structure is known as the *Pták sum*. In the following we will mention only the main properties of this structure. For any $a \in B \oplus L_1$ there exists an orthogonal partition 1 from B $\{c_1, \ldots, c_n\}$ and $a_1, \ldots, a_n \in L_1$ such that $\underset{\sim}{a} = \bigvee_{i=1}^{n} f(c_i) \wedge f_1(a_i)$. We can write $\underset{\sim}{a}$ as the "vector" $\underset{\sim}{a} = [(c_1, a_1), \ldots, (c_n, a_n)]$ and $f(c) = [(c, 1), (c^{\perp}, 0)], f_1(a) = [(1, a)]$.

2. HOMOMORPHISMS AND STATES

Proposition 2.1. Let L, A be a quantum logic, B be a Boolean algebra, and $B \oplus L$ be the Pták sum. Then a map $\gamma\colon B \oplus L \to A$ is a homomorphism iff there exist two homomorphisms h, g such that $h\colon B \to A, g\colon L \to A$, and $h(a) \leftrightarrow g(b)$, for any $a \in B$ and any $b \in L$ where $h = \gamma \circ f, g = \gamma \circ f_1$.

Proof. Let γ be a homomorphism. It is clear that $\gamma \circ f$, $\gamma \circ f_1$ are homomorphisms as well and, moreover, $\gamma \circ f\colon B \to A$, $\gamma \circ f_1\colon L \to A$. Let $a \in B$ and $b \in L$. We have

$$[(1, b)] = [(a, b), (a^{\perp}, b)] = (f(a) \wedge f_1(b)) \vee (f(a^{\perp}) \wedge f_1(b))$$

Then

$$\gamma[(1, b)] = \gamma[f(1) \wedge f_1(b)] = \gamma \circ f(1) \wedge \gamma \circ f_1(b) = \gamma \circ f_1(b)$$

but

$$\begin{aligned}\gamma[(a, b), (a^{\perp}, b)] &= \gamma(f(a) \wedge f_1(b) \vee f(a^{\perp}) \wedge f_1(b)) \\ &= \gamma \circ f(a) \wedge \gamma \circ f_1(b) \vee \gamma \circ f(a^{\perp}) \wedge \gamma \circ f_1(b)\end{aligned}$$

If we put

$$\gamma \circ f = h \qquad \text{and} \qquad \gamma \circ f_1 = g$$

then we get

$$g(b) = (h(a) \wedge g(b)) \vee (h(a^{\perp}) \wedge g(b))$$

This means that $g(b) \leftrightarrow h(a)$ holds for every $a \in B$ and $b \in L$.

Now we show the opposite implication. Let $h\colon B \to A$, $g\colon L \to A$ be such homomorphisms that $g(b) \leftrightarrow h(a)$ for any $b \in L$ and any $a \in B$. We show that a map τ from $B \oplus L \to A$ defined as

$$\tau([(a_1, b_1), \ldots, (a_n, b_n)]) = \bigvee_i h(a_i) \wedge g(b_i)$$

is the homomorphism.

Obviously $\tau([(1, 1)]) = 1$ and $\tau([(1, 0)]) = 0$. Without loss of generality it is enough to show the property of the supremum for the following elements: $\underline{b} = [(a, b_1), (a^\perp, b_2)]$, $\underline{c} = [(a, c_1), (a^\perp, c_2)]$, $b_i \leq c_i{}^\perp$, $i = 1, 2$.

$$\begin{aligned}\tau(\underline{c} \vee \underline{b}) &= \tau([(a, b_1 \vee c_1), (a^\perp, b_2 \vee c_2)]\\ &= h(a) \wedge g(b_1 \vee c_1) \vee h(a^\perp) \wedge g(b_2 \vee c_2)\\ &= h(a) \wedge (g(b_1) \vee g(c_1)) \vee (h(a^\perp) \wedge (g(b_2) \vee g(c_2))\\ &= h(a) \wedge g(b_1) \vee h(a) \wedge g(c_1) \vee h(a^\perp) \wedge g(b_2) \vee h(a^\perp) \wedge g(c_2)\\ &= h(a) \wedge g(b_1) \vee h(a^\perp) \wedge g(b_2) \vee h(a) \wedge g(c_1) \vee h(a) \wedge g(c_2)\\ &= \tau(\underline{b}) \vee \tau(\underline{c})\end{aligned}$$

From the known properties of a quantum logic it is clear that $r = s^\perp$ iff $r \vee s = 1$ and $r \leq s^\perp$. Let $[(a_1, b_1), \ldots, (a_n, b_n)] \in B \oplus L$. From the definition of the map τ we have

$$\tau([(a_1, b_1{}^\perp), \ldots, (a_n, b_n{}^\perp)]) = \bigvee_i h(a_i) \wedge g(b_i{}^\perp)$$

and

$$\tau([(a_1, b_1), \ldots, (a_n, b_n)]) = \bigvee_i h(a_i) \wedge g(b_i)$$

Evidently

$$\left(\bigvee_i (h(a_i) \wedge g(b_i))\right) \vee \left(\bigvee_i (h(a_i) \wedge g(b_i{}^\perp))\right) = 1$$

Let us put $r = \bigvee_i (h(a_i) \wedge g(b_i{}^\perp))$, $s = \bigvee_j (h(a_j) \wedge g(b_j))$. Then $s^\perp = \bigwedge_j (h(a_j)^\perp \vee g(b_j{}^\perp))$. Now we have for any $i, j \in \{1, \ldots, n\}$, $h(a_i) \wedge g(b_i{}^\perp) \leq h(a_i) \vee (b_i{}^\perp)$, and $h(a_i) \wedge g(b_i{}^\perp) \leq h(a_j)^\perp$ for any $i \neq j$. From this we can conclude that

$$\bigvee_i h(a_i) \wedge g(b_i{}^\perp) \leq \bigwedge_j (h(a_j)^\perp \vee g(b_j{}^\perp))$$

This means

$$\bigvee_i (h(a_i) \wedge g(b_i{}^\perp)) \leq \left(\bigvee_i (h(a_i) \wedge g(b_i))\right)^\perp$$

and then

$$\tau([(a_1, b_1{}^\perp), \ldots, (a_n, b_n{}^\perp)]) = \tau([(a_1, b_1), \ldots, (a_n, b_n)])^\perp$$

Thus we conclude that the map τ is the homomorphism from $B \oplus L$ to A. ∎

Proposition 2.2. The Pták sum $B \oplus L$ is a free product iff L is a Boolean algebra.

Proof. Let B, L be some Boolean algebras and $B \oplus L$ be the Pták sum. If we put $B \oplus L = \mathscr{L}$ and $u_1 = f$, $u_2 = f_1$ it is clear that the conditions (i), (ii) of the free product are fulfilled. Let A be any Boolean algebra. Let maps $\phi_1 : B \to A$, $\phi_2 : L \to A$ be homomorphisms. Hence $R(\phi_1)$, $R(\phi_2)$ are subsets of A, $\phi_1 \leftrightarrow \phi_2$. From Proposition 2.1 there exists a homomorphism $\phi : \mathscr{L} \to A$ such that $\phi_1 = \phi \circ u_1$, $\phi_2 = \phi \circ u_2$. From this it is clear that $B \oplus L$ is the free product.

Let $B \oplus L$ be a free product and A be any orthomodular lattice. Let $g : L \to A$ be any homomorphism. Let $a, b \in R(g)$ such that a is not compatible to b. Let us put $H = \{a, a^{\perp}, 0, 1\}$. And let h be a homomorphism from B on H such that there exists an element z with $h(z) = a$. It is clear that such a homomorphism exists and moreover $h(z)$ is not compatible to $b \in R(g)$. From Proposition 2.1 it follows that there does not exist a homomorphism γ from $B \oplus L$ to A. This means that $B \oplus L$ is not the free product so that A is the Boolean algebra. From the definition of the free product we have that A is any orthomodular lattice; then we can put $A = L$. Now we can conclude that L is the Boolean algebra. ■

Let L be a quantum logic. Let us denote $S(L)$ as the set of all states on L. Let B be a Boolean algebra. If $M_1 \subseteq S(B)$ and $M_2 \subseteq S(L)$, then $M_1 \times M_2 \subseteq S(B \oplus L)$ such that any $\mu \in M_1 \times M_2$ iff there exist $m_1 \in M_1$, $m_2 \in M_2$ with $\mu = (m_1, m_2)$.

Proposition 2.3. Let B and L be a Boolean algebra and a quantum logic, respectively, and $M_1 \subseteq S(B)$, $M_2 \subseteq S(L)$. Then $(B \oplus L_1, M_1 \times M_2)$ is qfs iff both (B, M_1), (L_1, M_2) are qfs.

Proof. Let $(B \oplus L, M_1 \times M_2)$ be qfs. Let us denote $\underline{B} = \{\underline{d} \in B \oplus L : \underline{d} = [(c, 1), (c^{\perp}, 0)]$ for $c \in B\}$. It is clear that $\underline{B}$ is a Boolean subalgebra $B \oplus L$ which is isomorphic to B and the restriction $M_1 \times M_2$ on $\underline{B}$ is isomorphic to M_1. From this we have (B, M_1) is qfs. Analogously, $\underline{L} = \{\underline{k} \in B \oplus L : \underline{k} = [(1, k)], k \in L\}$ is a sublogic of $B \oplus L$ and it is isomorphic to L. Moreover, the restriction $M_1 \times M_2$ on $\underline{L}$ is isomorphic to M_1. Then (L, S_1) is qfs.

Let (B, M_1), (L, M_2) be both qfs. Let $\{\mu \in M_1 \times M_2 : \mu(\underline{a}) = 1\} \subseteq \{\mu \in M_1 \times M_2 : \mu(\underline{b}) = 1\}$. We know there exist $c_1, \ldots, c_n \in B$ an orthogonal decomposition 1 in B, and $a_1, \ldots, a_n, b_1, \ldots, b_n \in L_1$ such that

$$\underline{a} = [(c_1, a_1), \ldots, (c_n, a_n)], \qquad \underline{b} = [(c_1, b_1), \ldots, (c_n, b_n)]$$

If $\mu(\underline{a}) = 1$, then there is exactly one $i \in \{1, \ldots, n\}$ such that $\mu(\underline{a}) = \sum_{j=1}^{n} m_0(c_j) m_1(a_j) = m_0(c_i) m_1(a_i) = 1$. From the assumption we have that

$\mu(\underline{a}) = 1$ implies $\mu(\underline{b}) = 1$. Then $m_1(b_i) = 1$. This means that $a_i \leq b_i$. Moreover, (B, M_1) is qfs, too, and from that for any $c \in B$, $c \neq 0$ there exists a state $m_0 \in M_1$ with $m_0(c) = 1$. Hence $a_i \leq b_i$ for all $i = 1, \ldots, n$ (where $c_i \neq 0$). ■

It is clear that $S(B) \times S(L) \subset S(B \oplus L)$. The following example shows that these sets do not have to be equal.

Example. Let $B = \{0, 1, a, a^\perp\}$, $C(L) = \{0, 1, b, b^\perp\}$ (L is a quantum logic). Let h be an isomorphism from B to $C(L)$ such that $h(a) = b$. Let $m \in S(L)$ such that $m(b) \neq 0, 1$ and $\mu = (m_h, m)$, $\alpha \in S(B \oplus L)$ such that $\alpha[(a_1, b_1), \ldots, (a_n, b_n)] = \sum m_i(h(a_i) \wedge b_i)$. It is clear that $\alpha/f(B) = \mu/f(B)$, $\alpha/f_1(L) = \mu/f_1(L)$, but $\alpha \neq \mu$. It is sufficient to take $\underline{c} = [(a, h(a^\perp)), (a^\perp, 0)]$.

Proposition 2.4. Let $B \oplus L$ be a Pták sum and h be a map from B to L. A map $\gamma: B \oplus L \to L$ which is defined as $\gamma([(c_1, a_1), \ldots, (c_n, a_n)]) = \bigvee_{i=1}^n h(c_i) \wedge a_i$ is a homomorphism iff h is the homomorphism from B to $C(L)$.

Proof. From the Proposition 2.1 we know that γ is a homomorphism iff the maps $\gamma \circ f, \gamma \circ f_1$ are homomorphisms and moreover $\gamma \circ f \leftrightarrow \gamma \circ f_1$. But $\gamma \circ f_1$ is the identical isomorphism from the assumption; then it is clear that $h = \gamma \circ f$ is the homomorphism whose range is the subset of $C(L)$. ■

Proposition 2.5. Let L be a quantum logic such that $C(L) \neq \{0, 1\}$ and B be a Boolean algebra. Let $m \in S(L)$ and h be a homomorphism from B to $C(L)$ such that there exist $c \in B$ with $m(h(c)) \neq 1, 0$. Then there exist two states $\alpha, \mu \in S(B \oplus L)$ such that $\alpha \neq \mu$ but $\alpha/f_1(L) = \mu/f_1(L), \alpha/f(B) = \mu/f(B)$.

Proof. From the previous proposition we know that there exists a homomorphism $\gamma: B \oplus L \to L$ such that $\gamma([(c_1, a_1), \ldots, (c_n, a_n)]) = \bigvee_i h(c_i) \wedge a_i$ for any $[(c_1, a_1), \ldots, (c_n, a_n)]$ from $B \oplus L$. If $m \in S(L)$, then m_γ is a state on $B \oplus L$ and

$$m_\gamma([(c_1, a_1), \ldots, (c_n, a_n)]) = \sum_{i=1}^n m(h(c_i) \wedge a_i)$$

On the other hand, m_h is a state on B and from the definition of the Pták sum there exists a state μ on $B \oplus L$ such that

$$\mu([(c_1, a_1), \ldots, (c_n, a_n)]) = \sum_{i=1}^n m(h(c_i))m(a_i)$$

Now it is enough to calculate

$$\mu([(c, h(c^{\perp})), (c^{\perp}, 0)]) = m(h(c))m(h(c^{\perp})) \neq 0$$

and

$$m_{\gamma}([(c, h(c^{\perp})), (c^{\perp}, 0)]) = m_{\gamma}(h(c) \wedge h(c^{\perp})) = 0$$

On the other hand, we have

$$\mu([(1, a)]) = m_{\gamma}([(1, a)])$$

$$\mu([(c, 1), (c^{\perp}, 0)]) = m_{\gamma}([(c, 1), (c^{\perp}, 0)]). \quad \blacksquare$$

Proposition 2.6. Let B and L be a Boolean algebra and a quantum logic, respectively, and let (L, M) be qfs. Here M is a convex set of states.

(i) If there exists a homomorphism h from B to L such that $R(h) \neq \{0, 1\}$, then there exist states μ, α from $S(B \oplus L)$ such that

$$\mu/f(B) = \alpha/f(B), \qquad \mu/f_1(L) = \alpha/f_1(L), \qquad \text{but } \mu \neq \alpha$$

(ii) $C(L) = \{0, 1\}$ iff for any homomorphism h from B to $C(L)$ and for any state m from M the relation

$$\sum_{i=1}^{n} m(h(c_i))m(a_i) = \sum_{i=1}^{n} m(h(c_i) \wedge a_i)$$

is satisfied, where $[(c_1, a_1), \ldots, (c_n, a_n)] \in B \oplus L$.

Proof. (ii) Let $C(L) = \{0, 1\}$; then for any homomorphism h from B to $C(L)$ we have $R(h) = \{0, 1\}$ and it is clear that for any state m on L we have

$$\sum_{i=1}^{n} m(h(c_i))m(a_i) = \sum_{i=1}^{n} m(h(c_i) \wedge a_i)$$

Let

$$\sum_{i=1}^{n} m(h(c_i))m(a_i) = \sum_{i=1}^{n} m(h(c_i) \wedge a_i)$$

for any state m from M and for any homomorphism from $C(L)$. Then for any $b \in B$ we have

$$m(h(b) \wedge h(b^{\perp})) = m(h(b))m(h(b^{\perp})) = 0$$

This means that $m(h(b)) = 1$ or 0.

Hence M is qfs, and for the convex set of states, we have $h(b) \in \{0, 1\}$ for any h. Then $C(L) = \{0, 1\}$.

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